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Existence of Bounded Nonoscillatory Solutions of Certain Nonlinear Neutral Delay Difference Equations

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Abstract- Non oscillation of a class of nonlinear neutral delay difference equations with positive and negative coefficients of the form

$$\Delta\left[y(n)+p(n)y(n-m)\right]+f_1(n)G_1(y(n-k_1))-f_2(n)G_2(y(n-k_2))=f(n) \quad \text{(E)}$$

is studied. We obtain the sufficient conditions for the existence of a bounded nonoscillatory solutions of (E) under the assumption

$$\sum_{n=0}^{\infty} f_i(\text{n})<\infty, \quad \text{for i}=1,2.$$

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I. INTRODUCTION

In this paper we study nonoscillation of a class of non homogeneous neutral delay difference equations with positive and negative coefficients of the form

$$\Delta\left[y(n)+p(n)y(n-m)\right]+f_1(n)G_1(y(n-k_1))-f_2(n)G_2(y(n-k_2))=f(n), \quad (1)$$

where $p(n)$, $f_1(n)$, $f_2(n)$, $f(n)$ are real valued functions defined on set $N(n_0)=\{n_0,n_0+1,n_0+2,......\}$, $n_0\geq 0$ such that $f_1(n)\geq 0$, $f_2(n)\geq 0$, $f(n)\geq 0$, G_1, G_2 are continuous real valued functions. G_1 and G_2 are non decreasing and $xG_i(x)>0$ for i=1, 2, $x\neq 0$, $n>0$ and $k_1,k_2,m\geq 0$ are integers, Δ is forward difference operator defined by equation,

$$\Delta x(n)=x(n+1)-x(n).$$

The corresponding differential equation to the difference equation (1) can be written as

$$\frac{d}{dt}\left[y(t)+p(t)y(t-\tau)\right]+f_1(t)G_1(y(t-\sigma_1))-f_2(t)G_2(y(t-\sigma_2))=f(t). \quad (2)$$

It is to remark that this equation when $f_2(t)\equiv 0$, $p(t)\equiv 0$ becomes a first order delay differential equation and we find numerous results regarding the solutions of this equations. we refer to [1],[2] and [3]and the references therein. Several researchers discussed nonoscillation and asymptotic behavior of solution of delay and neutral difference equations of first order. A close observation reveals that the study of difference equation is more or less similar to that of a differential equation. (See [4], [6], [7], [8] and [10]). In the recent papers [7, 8]. Parhi and Tripathy discussed oscillation and asymptotic behavior of solution of the equation

$$\Delta\left[y(n)-y(n-m)\right]+f_1(n)G_1(y(n-k_1))=f(n), \quad (3)$$

when $f_1(n)<0$ or when $f_1(n)>0$ under the condition

$$\sum_{n=0}^{\infty} f_1(n)=\infty.$$

It is predicted that the oscillation properties are not restricted to the sign of f_1.The motivation of the present work comes under two directions: firstly due to the above prediction and next due to the work in [5], where the authors considered the linear neutral differential equation

$$\frac{d}{dt}\left(\{y(t)+py(t-\tau)\}+f_1(t)y(t-\sigma_1)-f_2(t)y(t-\sigma_2)\right)=f(t), \quad (4)$$

where $p\neq -1$ is constant. The discrete analogue of equation (4) is a particular case of our equation (1). When $f(n)=0$ the existence of nonosillatory solutions was discussed in [9]. We consider various ranges of $p(n)$ and present the nonoscillation behavior of the solution.

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We recall that, by a solution of equation (1) on $N(n_0)$, we mean a real valued function $y(n)$ defined on $N(-\rho)=-\rho,-\rho+1,-\rho+2,-\rho+3.................$, which satisfies (1) for $n\geq n_0\geq 0,$ where $\rho=\max\{n,k_1,k_2\}$.

If $y(n)=A_n, \quad n=-\rho,-\rho+1....0,1,2,.....$ (5)

are given, then equation (1) admits a unique solution satisfying the initial condition (5). As is customary, a solution of (1) is said to be oscillate if for every integer $N>0$, there exists and $n\geq N$ such that $y(n)y(n+1)\leq 0.$ Otherwise, the solution is called nonoscillatory.

1. **W**e need the following the hypotheses in our discussion:

(H_1) : $G_i \in C(R,R)$, G_i is non decreasing for $i=1,2,\ R=(-\infty,\infty).$

(H_2) : $xG_i(x)>0$ for $x\neq 0$, $i=1,2.$

(H_3) : $G_i,\ i=1,2$ is Lipschitizain on the interval of the type [a, b], $0<a<b<\infty.$

(H_4) : $\sum_{n=0}^{\infty} f_i(n)<\infty$ for $i=1,2.$

Now we have the following:

THEOREM 2.1: Suppose that $0\leq p(n)\leq b_1<1$ and $(H_1)-(H_4)$ hold. If

$$\sum_{n=0}^{\infty} f\ (n)<\infty, \tag{6}$$

then there exists a bounded nonoscillatory solution of the equation (1).

Proof: Since G_1 and G_2 are Lipschitizain on the interval of the type [a, b], there exists

L_1, L_2 such that

$|G_1(y_1)-G_1(y_2)|\ \leq\ L_1|y_1-y_2|$ and

$|G_2(y_1)-G_2(y_2)|\ \leq\ L_2|y_1-y_2|$ for $y_1,y_2\in\left[\frac{1-b_1}{40},1\right]$.

Let $M_1=\max\{L_1,G_1(b_1)\}$, $M_2=\max\{L_2,G_2(b_1)\}$.

From hypotheses, we can find positive integer N_1 such that

$$\mathrm{M}_1\sum_{n=N_1}^{\infty} f_1(n)<\frac{9}{10\mathrm{M}_1}(1-b_1),$$

$$\mathrm{M}_2\sum_{n=N_1}^{\infty} f_2(n)<\frac{1}{20\mathrm{M}_2}(1-b_1),$$

(7)

$$\sum_{n=N_1}^{\infty} f\ (n)<\frac{1-b_1}{40}.$$

We consider $X=l_{\infty}^{N_1}$ be the Banach space of all real valued functions $x(n)$, $n\geq N_1$ with supremum norm

$$\|\ x\ \|=\sup\{\,|\ x(n)\ |:\ n\geq N_1\}.$$

And let

$$S=\left\{x\in X:\frac{1-b_1}{40}\leq x(n)\leq 1,\quad n\geq N_1\right\}.$$

It is easy to see that S is a complete metric space, where the metric is induced by norm on X.

For $y\in S$, we define the operator T as below:

$$Ty(n)=Ty(N_1+\rho),\qquad N_1\leq n\leq N_1+\rho.$$

(8)

$$=-p(n)y(n-m)+\frac{1+4b_1}{5}+\sum_{s=n}^{\infty} f_1(s)G_1(y(s-k_1))$$
$$-\sum_{s=n}^{\infty} f_2(s)G_2(y(s-k_2))-\sum_{s=n}^{\infty} f(s),$$

$$n\geq N_1+\rho$$

In view of hypotheses, we observe that

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$$Ty(n) < \frac{1+9b_1}{10} + M_1 \sum_{s=n}^{\infty} f_1(s)$$
$$< \frac{1+9b_1}{10} + \frac{9}{10}(1-b_1)$$

(from (7))

$$= 1,$$

and

$$Ty(n) > -b_1 + \frac{1+9b_1}{10} - M_2 \sum_{i=s}^{\infty} f_2(s) - \sum_{s=n}^{\infty} f(s)$$
$$> -b_1 + \frac{1+9b_1}{10} - \frac{1-b_1}{20} - \frac{1-b_1}{40},$$

(from (7))

$$= \frac{1-b_1}{40}, \qquad \text{for} \quad n \geq N_1 + \rho.$$

ie, $\frac{1-b_1}{40} < Ty(n) < 1.$

Consequently $Ty \in S$, that is $T: S \to S$.

Further for $x \in S$, consider

$$|Ty(n) - Tx(n)| \leq b_1 \|y-x\| + \frac{9}{10}(1-b_1)\|y-x\| + \frac{1-b_1}{20}\|y-x\|$$
$$= \left(b_1 + \frac{9}{10}(1-b_1) + \frac{1-b_1}{20}\right)\|y-x\|$$
$$= \frac{19+b_1}{20}\|y-x\|.$$

Thus $|Ty(n) - Tx(n)| \leq \frac{19+b_1}{20}\|y-x\|$ for every $x, y \in S$.

Hence T is contraction. By Banach fixed point theorem, T has a unique fixed point $y(n)$ in S, which will be a solution of equation (1) such that $\frac{1-b_1}{40} < Ty(n) < 1.$ we observe that this solution $y(n)$ is nonoscillatory and bounded.

Now for $-1 \leq b_2 \leq p(n) \leq 0,$ we have the following result:

THEOREM 2.2: Suppose $-1 \leq b_2 \leq p(n) \leq 0$ and $(H_1)-(H_4)$ hold. If

$$\sum_{n=0}^{\infty} f(n) < \infty,$$

then the equation (1) admits a bounded nonoscillatory solution.

Proof: We can choose a positive integer N_1 so large that

$$M_1 \sum_{n=N_1}^{\infty} f_1(n) < \frac{1+b_2}{5},$$
$$M_2 \sum_{n=N_1}^{\infty} f_2(n) < \frac{1+b_2}{10},$$
$$\sum_{n=N_1}^{\infty} f(n) < \frac{1+b_2}{20},$$

(9)

where M_1, M_2 and N_1 are same as in Theorem 2.1 on the interval $\left[\frac{1+b_2}{20}, 1\right].$

Let $X = l_{\infty}^{N_1}$ be the Banach space of all real valued functions $x(n)$, $n \geq N_1$ with supremum norm

$$\|x\| = \sup\{|x(n)| : n \geq N_1\}.$$

Let

$$S = \left\{ x \in X : \frac{1+b_2}{20} \leq x(n) \leq 1, \quad n \geq N_1 \right\}.$$

Again, it is easy to see that S is a complete metric space, where the metric is induced by norm on X.

Define a mapping T as

$$Ty(n) = Ty(N_1 + \rho), \quad N_1 \leq n \leq N_1 + \rho$$

(10)

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$$= -p(n)y(n-m) + \frac{1+b_2}{5} + \sum_{s=n}^{\infty} f_1(s)G_1(y(s-k_1))$$

$$- \sum_{s=n}^{\infty} f_2(s)G_2(y(s-k_2)) - \sum_{s=n}^{\infty} f\,(s), \quad n \geq N_1 + \rho.$$

For $y \in S$ and $n \geq N_1 + \rho$,

$$Ty(n) \leq -b_2 + \frac{1+b_2}{5} + M_1 \sum_{s=s}^{\infty} f_1(s)$$

$$< -b_2 + \frac{1+b_2}{5} + \frac{1+b_2}{5}$$

(from (9))

$$= \frac{2-3b_2}{5} < 1\,,$$

and

$$Ty(n) \geq \frac{1+b_2}{5} - \frac{1+b_2}{10} - \frac{1+b_2}{20}$$

$$= \frac{1+b_2}{20}.$$

(from (9))

Consequently $Ty \in S.$

For $x \in S$, we have

$$\left| \; Ty(n) - Tx(n) \; \right| \leq p(n)\|y-x\| + \frac{1+b_2}{5}\|y-x\| + \frac{1+b_2}{10}\|y-x\|$$

$$= \left(-b_2 + \frac{1+b_2}{5} + \frac{1+b_2}{10} \right)\|y-x\|$$

$$= \frac{3-7b_2}{10}\|y-x\|,$$

that is

$$\left| \; Ty(n) - Tx(n) \; \right| \leq \frac{3-7b_2}{10}\|y-x\|$$

so that T is a contraction on S. Therefore T has a unique fixed point y(n) in S, which will be nonoscillatory solution of equation (1) in the interval $\left[\frac{1+b_2}{20}, 1\right]$, which is also bounded.

Thus the proof is complete.

Remark 2.3: Following the lines of the proofs of Theorem 2.1 and Theorem 2.2, we can prove the existence of bounded nonoscillatory solution of the equation (1) when

(i) $-1 < \; p(n) \; < 1;$

(ii) $p(n) \; > 1 \;\; or \;\; p(n) < -1.$

However the details are omitted.

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